Math 347H: Fundamental Math (H) HOMEWORK 11 **Optional, not for turning in**

1. Prove that for each $x, r \in \mathbb{R}$ and a sequence $(x_n) \subseteq \mathbb{R}$, if $x_n \to x$ and $x > r$, then $\forall^\infty n$, $x_n > r$.

2. Let (X,d) be a metric space, let (x_n) be a sequence in it, and let $x \in X$. Prove that $x_n \to x$ if and only if $d(x_n, x) \to 0$.

3. Let (X,d) denote a metric space and let (x_n) denote a sequence in it. For each of the following choices of (X,d) and (x_n), determine whether (x_n) converges. Prove your answers.

(a) $X := \mathbb{R}$, $d(x,y) := |x-y|$, $x_n := \frac{n^2+1}{21n^2+3n+100}$.

(b) $X := \mathbb{R}$, $d(x,y) := |x-y|$, $x_n := \frac{n^2+1}{21n+100}$.

(c) $X := \mathbb{R}$, $d(x,y) := |x-y|$, $x_n := \frac{(-1)^n n}{n+1}$.

(d) $X := \mathbb{R}$, $d(x,y) := |x-y|$, $x_n := \frac{n^2}{2^n}$.

HINT: It is enough to prove that $2^n \geq n^3$.

(e) $X := \mathbb{R}^2$. For $x \in \mathbb{R}^2$, we write $x(i)$ for its i^{th} coordinate, so $x = \big(x(0), x(1)\big)$. For $x, y \in \mathbb{R}^2$, let $d(x,y) := d_\infty(x,y) := \max\{|x(0)-y(0)|, |x(1)-y(1)|\}$ and let $x_n := \left(\frac{1}{n+1}, n\right)$.

(f) $X := \mathbb{R}^2$, $d(x,y) := d_1(x,y) = |x(0)-y(0)| + |x(1)-y(1)|$, and $x_n := \left(\frac{n}{2n-1}, \frac{(-1)^n}{n+1}\right)$.

(g) $X := \mathbb{R}$ and $d(x,y) := |x-y|$. Let $1.d_1 d_2 d_3 \ldots$ be the decimal representation of $\sqrt{2}$. For each $n \in \mathbb{N}$, define $x_n := 1.d_1 d_2 \ldots d_n 000\ldots$, so $x_n \in \mathbb{Q}$.

(h) $X := \mathbb{Q}$, and d and x_n are the same as in the previous part.

4. Let $X := 2^{\mathbb{N}}$ and let d be the metric on X defined in class, namely: for $x, y \in 2^{\mathbb{N}}$, $d(x,y) := 2^{-\Delta(x,y)}$, where $\Delta(x,y)$ is the least index $i \in \mathbb{N}$ for which $x(i) \neq y(i)$.

(a) For each $n \in \mathbb{N}$, define an element $x_n \in 2^{\mathbb{N}}$ by $x_n := \underbrace{00\ldots0}_{n \text{ times}}111\ldots$. Prove that this sequence converges to the constant 0 sequence.

(b) More generally, given a sequence (x_n) in $2^{\mathbb{N}}$ and $x \in 2^{\mathbb{N}}$, prove the following necessary and sufficient condition for $x_n \to x$:

$$x_n \to x \iff \forall i \in \mathbb{N},\ \forall^\infty n \in \mathbb{N}\ x_n(i) = x(i).$$

CAUTION: In the condition on the right, the order of quantifiers really matters.

5. Prove that in any metric space, Cauchy sequences are bounded.

6. Let (x_n) be a bounded sequence of reals.

(a) Prove that if (x_n) is increasing, then it converges to its supremum $s := \sup\{x_n : n \in \mathbb{N}\}$.

(b) State and prove the analogous statement for decreasing (and bounded) sequences.

(c) Conclude the so-called Monotone Convergence Theorem: Bounded monotone (i.e. increasing or decreasing) sequence in $\mathbb{R}$ converge.